International Journal of Theoretical Physics, Vol. 32, No. 2, 1993

Classical Spin and Quantum Propagation

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Received March 13, 1992

We consider a classical Brownian motion model of diffusion in two spatial dimensions, where the Brownian particle moves on spiral paths. The classical spin does not change the propagator for the probability density for the position of the particle. However, the subdominant eigenvalues of the classical kernel are simply related to the dominant eigenvalues of the Feynman kernel for an analogous quantum system. The Feynman kernel can be extracted from the classical kernel by coupling to a spin angular momentum of the particle.

INTRODUCTION

Some recent work (Ord, 1992*a*) using spiral trajectories in a modification of the Feynman chessboard model (FCM) has suggested that some sort of intrinsic particle "spin" may be important in the phenomenon of quantum interference. In this paper we consider this possibility in the context of a classical diffusion model in two spatial dimensions.

In Section 1 we derive the diffusion equation in one dimension in two ways in order to gain familiarity with the role of "diffusive scaling" and transfer matrix methods in a well-known context.

In Section 2 we use the transfer matrix method to derive both the classical kernel and the Feynman propagator for classical walks with spin.

In Section 3 we discuss the results and speculate on the implications of possible generalizations to more realistic systems.

1. BROWNIAN MOTION AND DIFFUSION IN ONE DIMENSION

We consider here a simplified version of the Brownian motion, or random walk model of diffusion.

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Consider a particle on a discrete space-time lattice with respective lattice spacings δ and ε. Over each time interval ε the particle is required to move one step either to the left or right with equal probability. If $u(x, t)$ is the probability that the particle is at position x at time t, then conservation of probability is

$$u(x, t+\varepsilon) = \tfrac{1}{2}[u(x-\delta, t) + u(x+\delta, t)] \tag{1.1}$$

If we assume that u is a smooth function of x and t, a Taylor expansion of equation (1.1) equating lowest order terms gives

$$\frac{\partial u(x, t)}{\partial t}\varepsilon = \frac{\delta^2}{2}\frac{\partial^2 u(x, t)}{\partial x^2} \tag{1.2}$$

or

$$\frac{\partial u(x, t)}{\partial t} = \frac{\delta^2}{2\varepsilon}\frac{\partial^2 u(x, t)}{\partial x^2} \tag{1.3}$$

To obtain the diffusion equation from (1.3) we have to argue that as we refine the lattice spacing, the quantity $\delta^2/2\varepsilon$ must approach a constant D, say. That is, we assume that

$$\lim_{\delta \to 0} \frac{\delta^2}{2\varepsilon} = D \tag{1.4}$$

where D is a diffusion constant.

Physically this means that as we refine the lattice, the apparent speed of the particle goes up without limit. That is, the speed of the particle on the scale of δ is $s(\delta) \sim 2D/\delta$. We notice that if the speed of the particle on the scale of δ is $s(\delta)$, then the standard deviation of the velocity of the particle on that scale is just $\Delta v(\delta) = s(\delta)/2$, so that

$$\Delta v\, \delta \sim D \tag{1.5}$$

is asymptotically independent of scale.

This is the classical analog of the Heisenberg uncertainty principle, and its physical content in this context is easily understood. The Brownian particle moves very quickly on fine scales, and attempts to "localize" such a particle reveal a microscopic velocity inversely proportional to the length scale used.

In physical systems this Brownian motion velocity is bounded approximately by the speed of sound in the system, and on scales below the mean free path the "uncertainty relation" (1.5) is violated. On such fine scales the particle trajectory is a piecewise differentiable curve. Note that on intermediate scales between the size of the system and the mean free path the uncertainty relation (1.5) is a manifestation of the fact that the particle trajectory is a fractal with fractal dimension 2.

That is, equation (1.4) implies that the length of the particle's trajectory in unit time on the scale of δ is

$$L(\delta)=s(\delta)\sim 2\frac{D}{\delta} \tag{1.6}$$

But the length of a fractal curve with fractal dimension D_f is (Mandelbrot, 1977)

$$L(\delta)=L_0(\delta)^{1-D_f} \tag{1.7}$$

Thus the "uncertainty principle" here is simply generated by the fractal geometry of a random walk.

To see how "close" the random walk model comes to the Schrödinger equation, we will reconsider the above diffusion equation using a path integral approach.

Consider a space-time lattice with lattice spacings respectively δ and ε. We start a particle at the origin at $t=0$, and we construct the classical propagator $u(x, t)$ which is the probability that the particle arrives at (x, t) given that it started at $(0, 0)$. This propagator is just a sum over all classical paths from $(0, 0)$ to (x, t) (Figure 1).

Since the constraint that we count only paths that end at (x, t) is awkward, we instead "count" the Fourier transform of $u(x, t)$, namely $u(p, t)$. This may be done using a transfer matrix approach.

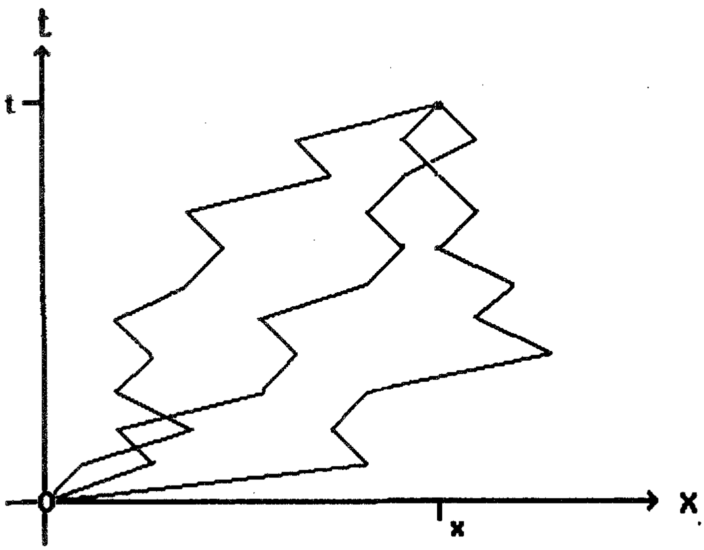

Fig. 1

The "diffusing" particle may be in one of two states at a given time. State 1 corresponds to moving in the $+x$ direction and state 2 corresponds to moving in the $-x$ direction. The transfer matrix for the transition between the two states is just

$$\mathbf{T}=\tfrac{1}{2}\begin{pmatrix} e^{-ip\delta} & 1 \\ 1 & e^{ip\delta} \end{pmatrix} \tag{1.8}$$

The physical interpretation of (1.8) (Ord, 1992*a*,*b*) is that both states are equally likely, and the exponents in the diagonal elements only serve to "count" the displacement of the walk.

For an N-step walk, $u(p, N\varepsilon)$ is

$$u(p, N\varepsilon)=\tfrac{1}{2}(1, 1)\mathbf{T}^{N}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.9}$$

assuming the walk is equally likely to start in either state one or two. Here the sums in the matrix product $\mathbf{T}_1^N$, i.e.,

$$[\mathbf{T}^N]_{ij}=\sum_{\sigma_2=1}^{2}\cdots\sum_{\sigma_{N-1}=1}^{2}\mathbf{T}_{i\sigma_1}\left[\prod_{k=1}^{N-2}\mathbf{T}_{\sigma_k\sigma_{k+1}}\right]\mathbf{T}_{\sigma_{N-1}j} \tag{1.10}$$

correspond to the "sum over paths" of the diffusing particle, each path consisting of a specific configuration $\{\sigma_1, \ldots, \sigma_{N-1}\}$.

Now calculating (1.9) in the limit as ε and δ go to zero is straightforward. To consider a diffusing particle with diffusion constant D, we require as in (1.4) that ε scales with δ^2. Thus, we wish to calculate

$$u(p, t)=\lim_{\delta\to 0}\ \tfrac{1}{2}(1, 1)(\mathbf{T}^{2Dt/\delta^2})\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.11}$$

Here $\mathbf{T}$ has a dominant eigenvalue, which to lowest order in δ is

$$\lambda_+=1-\frac{p^2\delta^2}{2} \tag{1.12}$$

with corresponding orthogonal projection operator

$$\mathbf{P}_+=\tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{1.13}$$

Thus

$$u(p, t) = \lim_{\delta \to 0} \left(1 - \frac{p^2 \delta^2}{2}\right)^{2Dt/\delta^2} = e^{-p^2 Dt} \tag{1.14}$$

To find the propagator $u(x, t)$, we transform this back to "position" space using

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(p, t) \exp(-ipx)\, dp$$
$$= (4\pi Dt)^{-1/2}\, e^{-x^2/4Dt} \tag{1.15}$$

This propagator clearly satisfies the diffusion equation and is normalized as a probability density with

$$\int_{-\infty}^{\infty} u(x, t)\, dx = 1 \tag{1.16}$$

Now the Feynman propagator for a "quantum" particle to propagate from the origin to (x, t) is (Feynman and Hibbs, 1965)

$$k(x, t) = \left(\frac{2\pi i\hbar t}{m}\right)^{-1/2} \exp\left(\frac{imx^2}{2\hbar t}\right) \tag{1.17}$$

and we note that this may be obtained from (1.15) by the replacement

$$D \leftarrow \frac{i\hbar}{2m} \tag{1.18}$$

This same replacement takes the diffusion equation over to the free-particle Schrödinger equation. However, although this analytic continuation is suggestive, it is precisely this step that lacks a classical counterpart since it converts the probability density (1.15) into the "probability amplitude" (1.17). This step will be considered closely in the next section. In the meantime we simply observe how the "sum over paths," equation (1.9), can be used to calculate (1.17).

The transfer matrix in equation (1.8) simply counts all paths with identical statistical weights. Let us consider a transfer matrix which "weights" each corner in the trajectory by a factor of i. That is, consider

$$\mathbf{T}_F = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-ip\delta} & i \\ i & e^{ip\delta} \end{pmatrix} \tag{1.19}$$

This matrix counts all configurations with weights that correspond to i^R, where R is the number of corners in the path.

Now the eigenvalues of (19) to lowest order in δ are

$$\lambda_{\pm}^{F}=e^{i\pi/4}\left(1\pm i\frac{p^2\delta^2}{2}\right) \tag{1.20}$$

Here λ^N does not converge, since there is an average phase shift of $\pi/4$ for each step in the walk; however, if we consider $N\to\infty$ through a sequence of integers which are 0 (mod 8), i.e.,

$$2Dt/\delta^2=0 \pmod 8 \tag{1.21}$$

then

$$\lim_{2Dt/\delta^2\to\infty} \lambda_{\pm}^{F}=e^{\mp ip^2Dt} \tag{1.22}$$

The projectors are

$$\mathbf{P}_{\pm}=\frac{1}{2}\begin{pmatrix} 1 & \mp 1 \\ \mp 1 & 1 \end{pmatrix} \tag{1.23}$$

and contraction to a scalar via (1.9) yields the "quantum" kernel

$$k(p,t)=e^{ip^2Dt} \tag{1.24}$$

This is the Fourier transform of (1.17) with the association

$$D\leftrightarrow\frac{\hbar}{2m} \tag{1.25}$$

Notice here that the analytic continuation which is necessary to transform the diffusive kernel (1.15) into the quantum kernel (1.17) has been accomplished above by the association of a phase angle with each path in an ensemble of paths between (0, 0) and (x, t). This ensemble is a necessary part of the formulation and it results in the usual problems in interpretation of (1.17). Namely, the propagator (1.17) has to describe the evolution of the state of a single particle using an ensemble of paths that the single particle is never "seen" to traverse. This is quite different from the classical propagator (1.15), in which the propagator describes only the state of one's knowledge of the system and requires only that the particle traverse one of the possible paths, with all paths equally likely.

The "phase rule" used above in the calculation of (1.24), which associates a phase of i to the power of the number of corners in the path with any given path, is "borrowed" from the Feynman chessboard model (Feynman and Hibbs, 1965; Gersch, 1981; Jacobson and Schulman, 1984; Ord, 1992*a*) of a relativistic particle.

2. SPIRAL DIFFUSION IN TWO DIMENSIONS

In a previous work (Ord, 1992*b*) spiral walks in space-time were used to obtain a Dirac propagator in such a way that a charge conservation principle was used to accomplish the analytic continuation needed to go from a classical partition function to a quantum propagator.

In this section we use spiral walks as the basis of a diffusive process in two dimensions. We set the problem up in such a way that we will later be able to "derive" a quantum propagator by considering the particle's "spin field." This will provide us with a Brownian-motion-like microscopic model for the quantum propagator.

Consider a particle moving on a two-dimensional square lattice with lattice spacing δ. The particle moves in discrete time at intervals of duration ε. The particle always moves one unit in both the x and y directions, but if it changes direction, it must always turn, say left. If we use the variables $p\delta$ and $q\delta$ to "count" displacement in the x and y directions, respectively, and we label states $(++)$. $(--)$, $(-+)$, $(+-)$ by $1, \ldots, 4$, respectively, then the transfer matrix is

$$T_s = \frac{1}{2}\begin{pmatrix} e^{-i(p+q)\varepsilon} & 0 & 0 & e^{-ip\delta} \\ 0 & e^{i(p+q)\varepsilon} & e^{+ip\delta} & 0 \\ e^{-q\delta} & 0 & e^{i(p-q)\varepsilon} & 0 \\ 0 & e^{+iq\delta} & 0 & e^{-i(p-q)\varepsilon} \end{pmatrix} \tag{2.1}$$

To second order in δ the eigenvalues of (2.1) are

$$\lambda_+^r = 1 - \frac{\delta^2}{2}(p^2+q^2) \tag{2.2}$$

$$\lambda_-^r = 0 \tag{2.3}$$

$$\lambda_\pm^c = e^{\pm i\pi/4}\left[1 \pm i\frac{\delta^2}{2}(p^2+q^2)\right] \tag{2.4}$$

The classical propagator $u(p, q, t)$ is obtained by starting the walk at $x = y = t = 0$ in any of the four possible directions each with probabilities of 1/4, and then adding contributions to all four final states. Thus the analog of (1.9) is

$$u(p, q, t) = \tfrac{1}{4}(1, 1, 1, 1)\mathbf{T}_s^N(1, 1, 1, 1)^T \tag{2.5}$$

In this calculation, as before, we scale ε as δ^2 so that $N = 2Dt/\delta^2$ and

$$\lim_{N\to\infty} (\lambda_r^+) = \lim_{\delta\to 0}\left(1 - \frac{\delta^2}{2}(p^2+q^2)\right)^{2Dt/\delta^2}$$
$$= e^{-Dt(p^2+q^2)} \tag{2.6}$$

The orthogonal projector $\mathbf{P}_+^r$ approaches

$$\mathbf{P}_+^r = \tfrac{1}{4}(1, 1, 1, 1)^T(1, 1, 1, 1) \tag{2.7}$$

and (2.3) becomes

$$u(p, q, t) = e^{-Dt(p^2+q^2)} \tag{2.8}$$

Taking the Fourier transform of this, we get

$$u(x, y, t) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(p, q, t)\, e^{-i(px+qy)}\, dp\, dq$$
$$= \frac{1}{4\pi Dt}\, e^{-(x^2+y^2)/4Dt} \tag{2.9}$$

which is the usual classical kernel.

Notice that we could have anticipated the result (2.9) on physical grounds. The spiral nature of the microscopic walks does not change the fact that at any given time, any of the four directions is equally likely. The fractal nature of the walk is preserved by requiring that $\varepsilon \sim \delta^2/2$, and there is nothing in the calculation of the kernel that could detect an intrinsic "spin." Thus, the kernel simply reflects the fact that the distribution of the endpoints of spiral walks is Gaussian, as it would be for simple random walks.

However, we notice the similarity of the complex eigenvalues (2.4) to the complex eigenvalues (1.20) associated with the Feynman propagator of the 1D walks [i.e., (1.17)]. The complex eigenvalues in this context are associated with the particle "spin" induced by spiral walks. This suggests that if we can somehow couple to the angular momentum of the diffusing particle, we might end up extracting a quantum propagator.

To see that the eigenvalues (2.3) bear more than just a superficial resemblance to the previous case (1.20), let us just proceed formally and consider

$$k^+(p, q, t) = \lim_{\delta\to 0}(\sqrt{2}\,\lambda_c^+)^{2Dt/\delta^2}$$
$$= e^{\mp iDt(p^2+q^2)} \tag{2.10}$$

where we have assumed that the limit is taken through a sequence of lattice spacings such that $2Dt/\delta^2 = 0 \pmod 8$.

Now (2.10) is just the Fourier transform of the Feynman propagator of a free particle of mass $\hbar/2D$. It is interesting to note that despite its "quantum" appearance, (2.10) is an entirely "classical" object. The transfer matrix (2.1) does nothing but count walks with real positive weights. There is no analytic continuation imposed by associating complex phases with the walks as there was in the derivation of (1.24). This suggests that we look for a physical interpretation of (2.10).

To extract a physical interpretation of the two complex eigenvalues $\lambda_{\pm}^{c}$, we consider a single walk coming into and out of a vertext (x, y) (Figure 2).

From the transfer matrix (2.1) we have assumed a symmetrical spiral walk in which the walker chooses to continue in its current direction or to turn left each with probability 1/2. Now the number of walks of N steps is just 2^N and the matrix product $\mathbf{T}_s^N$ simply counts all those walks, partitions them among powers of $e^{-iq\delta}$ and $e^{-ip\delta}$, and divides all terms by 2^N. This means that $\mathbf{T}_s$ is correctly normalized to calculate a probability. (Note that $\mathbf{T}_s$ evaluated at $\delta = 0$ is just a "transition matrix" for a Markov process.) Thus the classical propagator (2.7) is in fact correctly normalized as a probability density.

However, if we wish to extract information about the "spin" of these diffusing particles, we will have to consider observing not a scalar probability density, but a vector quantity, or complex number associated with the particle's velocity. Considering Figure 3, all paths entering (x, y) from

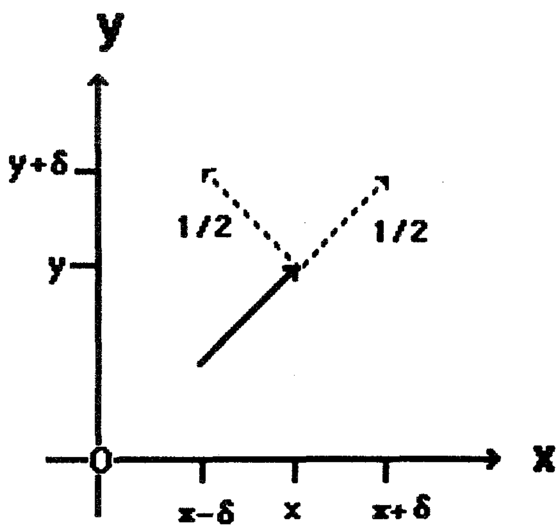

Fig. 2

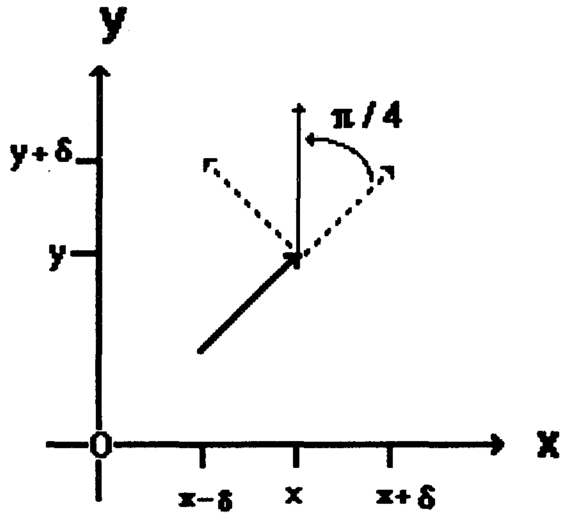

Fig. 3

$(x-\delta, y-\delta)$ generate new paths by exiting to either $(x+\delta, y+\delta)$ or $(x-\delta, y+\delta)$. The doubling of the number of paths at each step is exactly balanced in the calculation by introducing a weight of 1/2 for each choice. However, an alternative description of the symmetry of the walk is to say that the average direction of the walk in the xy plane changes by a factor of precisely $\pi/4$ at each step. Now if we wish to keep track of a complex number ϕ which describes the average orientation of the paths at each step, ϕ will be multiplied by $e^{i\pi/4}$ at each step. Furthermore, if we want to conserve the modulus of ϕ at each step, each branch will have to have a weight of $1/\sqrt{2}$ and not 1/2 as in the probability density case.

Figure 4 shows all walks of length 3 starting in state 1. To start with, $\phi = e^{i\pi/4}$ and $|\phi| = 1$. After three steps there are four separate path "ends," each with amplitude $e^{i\theta}$ for $\theta \in \{\pi/4, 3\pi/4, 5\pi/4\}$.

The sum of $|\phi|^2$ over all four terminal points is 1 and this description of a probability density agrees with simple diffusion (i.e., a probability of 1/4 for each terminal point). "Interference" effects will not actually happen until paths meet in opposite directions.

The "prescription" for finding ϕ, starting all walks in state 1, is then

$$\phi(q, p, t) = \lim_{\delta \to 0} (e^{i\pi/4})(1, -1, -i, i)(e^{i\pi/4}\sqrt{2}\mathrm{T}_s)^{2Dt/\delta^2}(1, 0, 0, 0)^T \qquad (2.11)$$

This may be evaluated in the usual way by an expansion of T_s in terms of its eigenvalues. To interpret (2.11), the first term in brackets is the phase of the first step in the walk; the subsequent row vector represents the relative

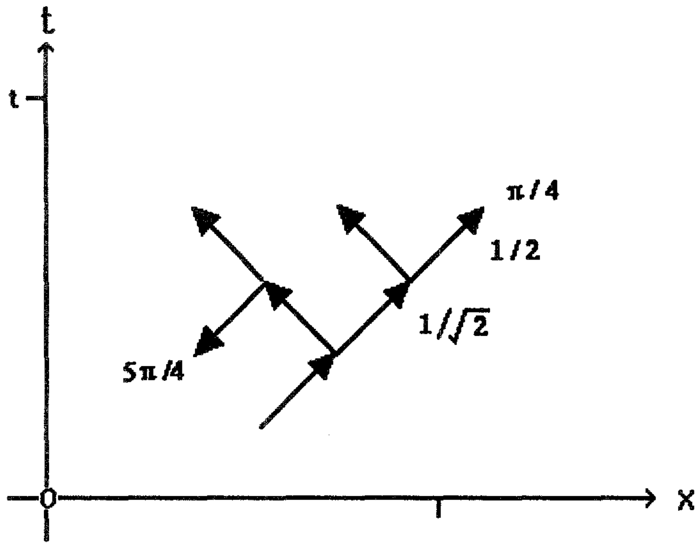

Fig. 4

phase contributions of the four states at the end of the walk. The factor $e^{i\pi/4}$ multiplying T changes the average phase by $\pi/4$ at each step of the walk. T_S just counts contributions from all walks, and the column vector "starts" all walks in state 1.

The contraction effected by left and right multiplication in (2.9) "selects" only the eigenvalue corresponding to λ_-^c in (2.3).

This leaves

$$\begin{aligned}\phi(q, p, t) &= \lim_{\delta \to 0} (e^{i\pi/4}) \left(1 - i\frac{\delta^2}{2}(p^2 + q^2)\right)^{2Dt/\delta^2} \\ &= e^{i\pi/4}\, e^{-iDt(p^2+q^2)}\end{aligned} \tag{2.12}$$

which is the Feynman propagator.

3. DISCUSSION

The appearance of (2.12) in connection with a classical spiral walk is something of a surprise. Noting that the transfer matrix (2.1) is a strictly "classical" object which only counts walks with real positive weights, we appear to have surreptitiously included an "analytic continuation" into the result (2.12). Retracing our steps in the calculation, the "suspicious" i in the exponent in (2.10) is a direct consequence of the complex eigenvalues $\lambda_\pm^c$ of

$\mathbf{T}_s$ in (2.4). This in turn is a consequence of the spiral nature of the walks and the resulting periodicity of $\mathbf{T}_s$. The classical "spin" of the particle has in fact "built in" the analytic continuation for us.

If the result (2.12) is not just a fortuitous accident in the case of a free particle in two dimensions, the result strongly suggests that we look to special relativity for answers concerning questions on the origin of quantum interference. The reason for this is as follows. $\phi(q, p, t)$ of equation (2.11) is an average over an ensemble of paths. The only way that $|\phi|^2$ can be construed as a probability density for a single particle is if we are forced to measure $\phi(q, p, t)$ as an average field created by a single-particle trajectory. This we are indeed forced to do if we consider diffusive scaling as in (1.4). In this case the fractal nature of the trajectory precludes the observation of anything but averages over trajectories of infinite length. In this sense the "quantum" aspect of the propagator is intimately associated with the "classical" uncertainty principle (1.5) and the resulting "unphysical" infinite velocities that result from a maintenance of this principle on all scales.

Special relativity implies that there will be a cutoff scale (the Compton wavelength) at which the above "diffusive scaling" breaks down. How the uncertainty principle is maintained below this scale must in the end tell us whether or not ϕ is a sensible object to associate with a single-particle trajectory. Previous work suggests that it may well be a sensible single-particle object; however, a reasonable proof of this fact awaits a fully relativistically correct version of a microscopic model similar to the one above.

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